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Impluse-free output regulation of singular nonlinear systems

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This paper addresses the output regulation problem for the class of singular nonlinear systems. A generalized version of the centre manifold theorem that applies to singular nonlinear systems is established first. Then necessary and sufficient conditions are given for the solvability of the output regulation problem. The work extends the existing results of output regulation problem for singular linear systems or *normal* nonlinear systems to the singular nonlinear systems.

1. Introduction

Singular systems are dynamical systems whose behaviours are governed by both differential equations and algebraic equations. Such systems arise in electrical networks, power systems, large-scale systems, etc. Over the past two decades, there has been extensive study on singular systems encompassing such issues as solvability (Aplevich 1981, Campbell and Griepentrog 1995), controllability and observability (Campbell 1980, Verghese *et al.* 1981, Yip and Sincovec 1981, Cobb 1984, Lewis 1985, 1986, Dai 1989), pole assignment and the elimination of impulsive behaviour (Cobb 1981, Cheng and Zhang 1986, Dai 1989), LQG control (Pandolfi 1981, Cobb 1983, Bender and Laub 1987 a, b, Cheng *et al.* 1988, Jonckheere 1988), output regulation (Lin and Dai 1996), and input–output decoupling (Cheng and Zhang 1986, Liu *et al.* 1996), to name just a few. Such efforts not only have extended a substantial portion of the research results on *normal* systems to this more general class of dynamic systems, but also led to many practical applications involving economics (Luenberger 1977 a, b), power systems (Hill and Mareels 1990), robot control (McClamroch and Wang 1988) and so on.

In this paper, we will study an output regulation problem for singular nonlinear systems. Roughly, by output regulation problem, we mean the design of control laws for a plant so that the output of the closed-loop system is able to asymptotically track a class of reference inputs and reject a class of disturbances. Both the disturbance and reference are generated by an autonomous differential equation termed exosystem. This problem was first studied in the 1970s for the class of *normal* linear systems in the form of a servomechanism problem or output regulation problem (Davison 1976, Francis and Wonham 1976, Francis 1977). The same problem for the *normal* nonlinear systems has also been pursued since the late 1980s (Isidori and Byrnes 1990, Huang and Rugh 1990, 1992). In particular, the work of Isidori and Byrnes in their award-winning paper (Isidori and Byrnes 1990) has led to the discovery of the salient regulator equations that have become the cornerstone for the

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study of nonlinear output regulation problems. The study of this problem for singular systems, however, has long been limited to the class of linear systems (Dai 1989, Lin and Dai 1996), and it was only recently that a clear-cut solution was obtained by Lin and Dai (1996).

Inspired mainly by the work of Isidori and Byrnes (1990) and Lin and Dai (1996), we have tried to tackle the output regulation problem for singular nonlinear systems. The outcome of our research has led to the present paper. The highlight of our research along with the organization of the rest of this paper is given as follows. In §2, we present a precise formulation of the output regulation problem for the singular nonlinear systems in such a way that it is general enough to include the output regulation problem for linear singular systems or *normal* nonlinear systems as a special case, and yet tight enough to be able to eliminate, in the resulting closed-loop system, the annoying impulsive behaviour. In §3, we will establish a generalized version of the centre manifold theorem that applies to singular nonlinear systems. This generalization, in addition to laying down the foundation for studying the output regulation problem for singular nonlinear systems, exhibits an important property of singular nonlinear systems, namely, the time response of a singular nonlinear system is impulse-free provided the linearization of the singular nonlinear system is *strongly stable*. Armed with the generalized version of the centre manifold theorem, we establish, in §4, the solvability conditions of our problem under either state feedback or *singular* output feedback. It turns out that the solvability of the output regulation problem is tied to the solvability of a set of singular partial differential equations and algebraic equations that can be viewed as the singular analogue of the regulator equations associated with *normal* nonlinear systems (Isidori and Byrnes 1990). In §5, we further address the important issue of looking for *normal* output feedback control to achieve output regulation in singular nonlinear systems. This issue is interesting since it is difficult to realize a singular controller physically. Section 6 presents a result that leads to an approximation method for solving the singular regulator equations. Section 7 closes this paper with some remarks.

2. Problem Formulation

Consider the plant described by

$$\left.\begin{aligned} E\dot{x}(t) &= f(x(t), u(t), w(t)), \qquad x(0) = x_0 \\ e(t) &= h(x(t), w(t)), \quad t \geq 0 \end{aligned}\right\} \tag{1}$$

and an exosystem described by

$$\dot{w}(t) = s(w(t)), \qquad w(0) = w_0 \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the plant state, $u(t) \in \mathbb{R}^m$ the plant input, $e(t) \in \mathbb{R}^p$ the plant output representing the tracking error, $w(t) \in \mathbb{R}^q$ the exogenous signal representing the disturbance and/or the reference input, and $E \in \mathbb{R}^{n\times n}$ a singular constant matrix.

We will focus on two classes of control laws, namely,

(1) State feedback control described by

$$u(t) = k(x(t), w(t)) \tag{3}$$

and

(2) Output feedback control described by

$$\left.\begin{aligned} u(t) &= \beta(z(t), e(t)) \\ E_z\dot{z}(t) &= g(z(t), e(t)) \end{aligned}\right\} \tag{4}$$

where $z(t)$ is the compensator state vector of dimension n_z to be specified later, and $E_z \in \mathbb{R}^{n_c \times n_c}$ is constant.

Equation (4) is said to be a *normal* controller if E_z is an identity matrix. The closed-loop system composed of plant (1), (2) and control law (3) or (4) can be put into the following form:

$$\left.\begin{aligned} E_c\dot{x}_c(t) &= f_c(x_c(t), w(t)), \qquad x_c(0) = x_{c0} \\ \dot{w}(t) &= s(w(t)), \qquad w(0) = w_0 \\ e(t) &= h_c(x_c(t), w(t)) \end{aligned}\right\} \tag{5}$$

where for the state feedback case, $x_c = x$, $E_c = E$, $f_c(x, w) = f(x, k(x, w), w)$, and $h_c(x_c, w) = h(x, w)$, and for the output feedback case, $x_c = \begin{bmatrix} x^T & z^T \end{bmatrix}$, and

$$E_c = \begin{bmatrix} E & 0 \\ 0 & E_z \end{bmatrix}, \qquad f_c(x_c, w) = \begin{bmatrix} f(x, \beta(z, h(x, w)), w) \\ g(z, h(x, w)) \end{bmatrix}, \qquad h_c(x_c, w) = h(x, w) \tag{6}$$

For simplicity, all the functions involved in this setup are assumed to be sufficiently smooth and defined globally on the appropriate Euclidean spaces, with the value zero at the respective origins. Our results will be stated locally in terms of an open neighbourhood W of the origin in $\mathbb{R}^q$, and we implicitly permit W to be made smaller to accommodate subsequent local arguments. We denote the dimension of x_c by n_c with the understanding that $n_c = n$ for the state feedback case and $n_c = n + n_z$ for the output feedback case.

Output regulation problem: Find a control law (state feedback or output feedback) such that the closed-loop system (5) has the following two properties:

R1: The linearization of

$$E_c\dot{x}_c = f_c(x_c, 0) \tag{7}$$

is strongly stable in the sense to be described in Remark 1.

R2: The trajectories starting from all sufficiently small initial state (x_{c0}, w_0) satisfies

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} h_c(x_c(t), w(t)) = 0 \tag{8}$$

Remark 1: Let the linearization of (7) be described by

$$E_c\dot{x}_c = A_c x_c \tag{9}$$

Then (E_c, A_c) is said to be *strongly stable* if $\deg(\det(sE_c - A_c)) = \operatorname{rank}(E_c)$, and $\sigma(E_c, A_c) \in C^-$ where $\sigma(E_c, A_c) \triangleq \{s \mid \det(sE_c - A_c) = 0\}$ (Dai 1989). Clearly, when $E_c = I$, those two requirements reduce to what were imposed on the output regulation problem for the *normal* system as studied by Isidori and Byrnes (1990). On the other hand, Requirement R1 is somehow stronger than its linear version studied by Lin and Dai (1996) where only the closed-loop system stability, i.e. $\sigma(E_c, A_c) \in C^-$, was required. This strengthening is made for both technical and practical considerations. In terms of practicality, the additional condition $\deg(\det(sE_c - A_c)) = \operatorname{rank}(E_c)$ guarantees, as will be made clear in Remark 3 later, that the response of the closed-loop system (5) is impulse-free, a desirable property by all practical engineering systems. In terms of technicality, requirement R1 guarantees, as will be shown in Theorem 1, that the closed-loop system will induce a stable centre manifold at the origin of R^{n_c+q} that is crucial for the fulfilment of R2.

Many of our results will rely on the properties of the linear approximation of the plant and the exosystem. Therefore, we introduce the notation

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0,u=0,w=0}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{x=0,u=0,w=0}, \quad P = \left.\frac{\partial f}{\partial w}\right|_{x=0,u=0,w=0}$$

$$C = \left.\frac{\partial h}{\partial x}\right|_{x=0,w=0}, \quad Q = \left.\frac{\partial h}{\partial w}\right|_{x=0,w=0}, \quad S = \left.\frac{\partial s}{\partial w}\right|_{w=0}$$

Thus the linear approximation of the plant and the exosystem at the origin is described by

$$\left.\begin{aligned} E\dot{x} &= Ax + Bu + Pw, \\ e &= Cx + Qw, \\ \dot{w} &= Sw, \end{aligned}\right\} \tag{10}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $P \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{p\times n}$, $Q \in \mathbb{R}^{p\times q}$, and $S \in \mathbb{R}^{q\times q}$.

Now we are ready to list the following hypotheses:

(H1): $w = 0$ is a stable equilibrium of the exosystem, and there exists a neighborhood W in the origin of R^q with the property that each initial condition $w_0 \in W$ is stable in the sense of Poisson.

(H2): (E, A, B) is strongly stabilizable, i.e. there exists a matrix $K \in \mathbb{R}^{m\times n}$ such that $(E, A + BK)$ is strongly stable.

(H3):

$$\left(\begin{bmatrix} E & 0 \\ 0 & I_q \end{bmatrix}, \begin{bmatrix} A & P \\ 0 & S \end{bmatrix}, \begin{bmatrix} C & Q \end{bmatrix} \right) \quad \text{is strongly detectable,}$$

i.e. there exist matrices $G_1 \in \mathbb{R}^{n\times p}$ and $G_2 \in \mathbb{R}^{s\times p}$ such that

$$\left(\begin{bmatrix} E & 0 \\ 0 & I_q \end{bmatrix}, \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} - \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \begin{bmatrix} C & Q \end{bmatrix} \right) \quad \text{is strongly stable}$$

Remark 2: Hypothesis (H1) is a standard assumption introduced by Isidori and Byrnes in order to invoke the centre manifold theorem (Isidori and Byrnes 1990). Hypotheses (H2) and (H3) are made to ensure the fulfilment of R1 by state feedback or output feedback. We note that Hypotheses (H2) and (H3) are somewhat stronger than those used by Lin and Dai (1996) due to the enhancement of our requirement R1 over the corresponding linear version as already commented in Remark 1. Also, we note that, in the special case where $E = I$, Hypotheses (H2) and

(H3) reduce to exactly the same ones assumed by Isidori and Byrnes for the output regulation problem of the *normal* systems (Isidori and Byrnes 1990).

3. A generalized version of the centre manifold theorem

It is known that the centre manifold theorem plays a key role in solving the output regulation problem for *normal* nonlinear systems (Isidori and Byrnes 1990). In this section, we will establish a generalized version of the centre manifold theorem that applies to the class of the singular nonlinear systems described by

$$\left.\begin{aligned} E_c\dot{x}_c &= f_c(x_c, w) \\ \dot{w} &= s(w) \end{aligned}\right\} \tag{11}$$

Theorem 1: *Consider the system* (11). *Assume* H1, *and suppose the linear approximation at the origin of* $E_c\dot{x}_c = f_c(x_c, 0)$ *is strongly stable. Then*

(1) *there exists a sufficiently smooth function* $\mathbf{x}_c(w)$ *locally defined around the origin of* R^q *satisfying* $\mathbf{x}_c(0) = 0$, *and*

$$E_c\frac{\partial \mathbf{x}_c(w)}{\partial w}s(w) = f_c(\mathbf{x}_c(w), w) \tag{12}$$

(2) *for all sufficiently small* x_{c0} *and* w_0, *the solution of* (11) *denoted by* $(x_c(t), w(t))$, *is sufficiently smooth for all* $t > 0$, *and satisfies,*

$$\lim_{t\to\infty}\left[x_c(t) - \mathbf{x}_c(w(t))\right] = 0 \tag{13}$$

Proof: Part (1): Rewrite the system (11) into the form

$$E_c\dot{x}_c = A_c x_c + B_c w + \phi(x_c, w) \tag{14}$$

$$\dot{w} = Sw + \psi(w) \tag{15}$$

where $\phi(x_c, w)$ and $\psi(w)$ vanish at their origins together with their first-order derivatives.

Assume $\operatorname{rank}(E_c) = r$. Then there exist two nonsingular matrices M_1 and M_2 such that

$$M_1 E_c M_2 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

where I_r denotes the $r \times r$ identity matrix. Let

$$\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = M_1 A_c M_2, \quad \text{with } \bar{A}_{11} \in \mathbb{R}^{r\times r}$$

$$M_1 B_c \triangleq \begin{bmatrix} B_1^{\mathrm{T}} & B_2^{\mathrm{T}} \end{bmatrix}, \quad \text{with } B_1 \in \mathbb{R}^{r\times q}$$

$$\bar{x}_c = M_2^{-1} x_c \triangleq \begin{bmatrix} \bar{x}_1^{\mathrm{T}} & \bar{x}_2^{\mathrm{T}} \end{bmatrix} \quad \text{with } \bar{x}_1 \in \mathbb{R}^r$$

$$M_1\phi(x, w) \triangleq \begin{bmatrix} \bar{\phi}_1^{\mathrm{T}}(\bar{x}_1, \bar{x}_2, w) & \bar{\phi}_2^{\mathrm{T}}(\bar{x}_1, \bar{x}_2, w) \end{bmatrix} \quad \text{with } \bar{\phi}_1 \in \mathbb{R}^r$$

Then, premultiplying by M_1 both sides of (14) gives

$$\dot{\bar{x}}_1 = \bar{A}_{11}\bar{x}_1 + \bar{A}_{12}\bar{x}_2 + \bar{B}_1 w + \bar{\phi}_1(\bar{x}_1, \bar{x}_2, w) \tag{16}$$

$$0 = \bar{A}_{21}\bar{x}_1 + \bar{A}_{22}\bar{x}_2 + \bar{B}_2 w + \bar{\phi}_2(\bar{x}_1, \bar{x}_2, w) \tag{17}$$

where $\bar{\phi}_1(\bar{x}_1, \bar{x}_2, 0)$ and $\bar{\phi}_2(\bar{x}_1, \bar{x}_2, 0)$ vanish at $(\bar{x}_1, \bar{x}_2) = 0$ together with their first-order derivatives.

Notice that $\deg(\det(sE_c - A_c)) = \deg(\det(M_1(sE_c - A_c)M_2))$ and

$$\det(M_1(sE_c - A_c)M_2) = \det\left(\begin{bmatrix} sI_r - \bar{A}_{11} & -\bar{A}_{12} \\ -\bar{A}_{21} & -\bar{A}_{22} \end{bmatrix}\right) = \det(-\bar{A}_{22})s^r + b(s)$$

where $b(s)$ is a polynomial in s of degree smaller than r. It follows from the assumption $\deg(\det(sE_c - A_c)) = \operatorname{rank}(E_c) = r$ that $\det(-\bar{A}_{22}) \neq 0$. Thus, $\bar{A}_{22}$ is non-singular. By the Implicit Function Theorem, there exists a unique sufficiently smooth function $\bar{x}_2 = \alpha(\bar{x}_1, w)$ satisfying $\alpha(0, 0) = 0$, and

$$0 = \bar{A}_{21}\bar{x}_1 + \bar{A}_{22}\alpha(\bar{x}_1, w) + \bar{B}_2 w + \bar{\phi}_2(\bar{x}_1, \alpha(\bar{x}_1, w), w) \tag{18}$$

Furthermore,

$$\left.\frac{\partial \alpha(\bar{x}_1, w)}{\partial \bar{x}_1}\right|_{\substack{\bar{x}_1 = 0 \\ w = 0}} = -\bar{A}_{22}^{-1}\bar{A}_{21}$$

Substituting $\bar{x}_2 = \alpha(\bar{x}_1, w)$ into (16) gives

$$\dot{\bar{x}}_1 = (\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}\bar{A}_{21})\bar{x}_1 + \bar{B}_1 w + \bar{\phi}_3(\bar{x}_1, w) \tag{19}$$

where $\bar{\phi}_3(\bar{x}_1, 0)$ vanishes at $\bar{x}_1 = 0$ with its first-order derivatives.

By a straightforward calculation, we get

$$\det(sE_c - A_c) = \det(M_1^{-1}M_2^{-1}) \cdot \det\left(\begin{bmatrix} sI_r - \bar{A}_{11} & -\bar{A}_{12} \\ -\bar{A}_{21} & -\bar{A}_{22} \end{bmatrix}\right)$$

$$= \det(M_1^{-1}M_2^{-1})\det(-\bar{A}_{22})\det(sI_r - (\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}))$$

Thus $\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}$ is stable since by assumption $\sigma(E_c, A_c) \in C^-$.

Now consider the following *normal* system

$$\left.\begin{aligned} \dot{\bar{x}}_1 &= \bar{f}_1(\bar{x}_1, w) = (\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}\bar{A}_{21})\bar{x}_1 + \bar{B}_1 w + \bar{\phi}_3(\bar{x}_1, w) \\ \dot{w} &= s(w) \end{aligned}\right\} \tag{20}$$

Since all the eigenvalues of $(\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}\bar{A}_{21})$ have negative real parts, and all the eigenvalues of S have zero real parts by H1, by the centre manifold theorem (Carr 1981, Isidori and Byrne 1990), this system has a stable centre manifold locally defined at the origin of R^{r+q}, or, what is the same, there exist a sufficiently smooth function $\bar{x}_1 = \pi(w)$ locally defined around the origin of R^q satisfying $\pi(0) = 0$, and is such that

$$\frac{\partial \pi(w)}{\partial w}s(w) = (\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}\bar{A}_{21})\pi(w) + \bar{B}_1 w + \bar{\phi}_3(\pi(w), w) \tag{21}$$

Moreover, there exist positive constants M and σ such that, for all sufficiently small $\bar{x}_1(0)$, and w_0, the solution of (20) satisfies

$$\left\|\bar{x}_1(t) - \pi(w(t))\right\| \leq M e^{-\sigma t}\left\|\bar{x}_1(0) - \pi(w(0))\right\|, \quad t \geq 0 \tag{22}$$

Let

$$\mathbf{x}_c(w) = M_2 \begin{bmatrix} \pi(w) \\ \alpha(\pi(w), w) \end{bmatrix} \tag{23}$$

Then it is ready to verify, using (18) and (21), that (23) satisfies (12).

Part (2): In terms of the solution of (20), we can define

$$x_c(t) = \begin{cases} M_2 \begin{bmatrix} \bar{x}_1(t) \\ \alpha(\bar{x}_1(t), w(t)) \end{bmatrix}, & t > 0 \\ x_{c0}, & t = 0 \end{cases} \tag{24}$$

Clearly, for $t > 0$, $x_c(t)$ is sufficiently smooth and $(x_c(t), w(t))$ satisfies (11). Moreover,

$$\lim_{t\to\infty}\left[x_c(t) - \mathbf{x}_c(w(t))\right]$$

$$= M_2 \begin{bmatrix} \lim_{t\to\infty}(\bar{x}_1(t) - \pi(w(t))) \\ \alpha\left(\lim_{t\to\infty}\bar{x}_1(t) - \pi(w(t))\right) + \pi(w(t), w(t)) - \alpha(\pi(w(t)), w(t)) \end{bmatrix} = 0 \tag{25}$$

since $\alpha(\cdot,\cdot)$ is continuous. □

Remark 3: It is known that the response of a strongly stable linear system is impulse free. This nice property is also retained for the singular nonlinear system described by (11) when the linearization of $E_c\dot{x}_c = f_c(x_c, 0)$ is strongly stable. This is evident from the explicit expression given by (24). However, as opposed to the *normal* system, the response $x_c(t)$ may be discontinuous at $t = 0$. The magnitude of the discontinuity of $x_c(t)$ as given by (24) can be calculated as follows. Let

$$x_{c0}(0+) = \lim_{t\to 0+} x_c(t)$$

and

$$M_2^{-1} = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}$$

where $\Gamma_1 \in R^{r\times n_c}$. Then the magnitude of the discontinuity of $x_c(t)$ at $t = 0$ is

$$x_c(0+) - x_{c0} = M_2 \begin{bmatrix} 0 \\ \alpha(\Gamma_1 x_{c0}, w_0) - \Gamma_2 x_{c0} \end{bmatrix}$$

Remark 4: A geometric interpretation of Theorem 1 can be given as follows. Let $x_a = \begin{bmatrix} x_c^\tau & w^\tau \end{bmatrix}$, and rewrite system (11)

$$E_a\dot{x}_a = \begin{bmatrix} f_c(x_c, w) \\ s(w) \end{bmatrix} \triangleq f_a(x_a) \tag{26}$$

where $E_a = \text{diag}\ (E_c, I_q)$. Then equations (12) and (2) are equivalent to

$$E_a \frac{\partial \mathbf{x}_a(w)}{\partial w} s(w) = f_a(\mathbf{x}_a(w)) \tag{27}$$

Thus the manifold defined by $\mathbf{x}_a(w) \triangleq \begin{bmatrix} \mathbf{x}_c(w)^\tau & w^\tau \end{bmatrix}$ for $w \in W$ is a locally invariant manifold for the singular system (26). What is more, we can show that $\mathbf{x}_a(w)$ is actually a centre manifold for the system (26) in a meaningful sense. In fact, denote the Jacobian matrices of $f_a(x_a)$ and $\mathbf{x}_a(w)$ at their origins by A_a and X_a, respectively. It is not difficult to verify, by linearizing (27), that

$$E_a X_a S = A_a X_a \tag{28}$$

Since $\sigma(E_a, A_a) = \sigma(E_c, A_c) \cup \sigma(I_q, S)$, and the matrix S has only zero-real-part eigenvalues, X_a is composed by those generalized eigenvectors of the pair (E_a, A_a) that are associated with all those generalized eigenvalues of the pair (E_a, A_a) whose real parts are zero. In other words, the eigenspace of (E_a, A_a) associated with the eigenvalues of (I_q, S) is the tangent space to the manifold $\mathbf{x}_a(w)$ at $x_a = 0$. Thus, the manifold $\mathbf{x}_a(w)$ can be reasonably called as the local centre manifold of the system (26) passing through $x_a = 0$.

4. Solvability of the output regulation problem

We are now ready to tackle the solvability of the output regulation problem. We begin by translating the requirement R2 into an algebraic constraint on the function $\mathbf{x}_c(w)$ defined in Theorem 1.

Lemma 1: *Assume* H1, *and suppose there exists a control law (state or output feedback) such that the closed-loop system* (5) *satisfies* R1, *then the closed-loop system also satisfies* R2 *if and only if there exists a sufficiently smooth function* $\mathbf{x}_c(w)$ *locally defined in* $w \in W$ *with* $\mathbf{x}_c(0) = 0$ *such that*

$$E_c \frac{\partial \mathbf{x}_c(w)}{\partial w} s(w) = f_c(\mathbf{x}_c(w), w) \tag{29}$$

$$0 = h_c(\mathbf{x}_c(w), w) \tag{30}$$

Proof:

(Sufficiency) Assume (29) and (30) hold for some $\mathbf{x}_c(w)$. Then by Theorem 1, for all sufficiently small x_{c0}, and w_0, the solution of (11) satisfies

$$\lim_{t\to\infty}\left[x_c(t) - \mathbf{x}_c(w(t))\right] = 0 \tag{31}$$

It follows from the continuity of $h(\cdot,\cdot)$ as well as (30) and (31) that

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty}\left[h_c(x_c(t), w(t)) - h(\mathbf{x}_c(w(t), w(t)))\right] = 0 \tag{32}$$

(Necessity) Since the closed-loop system (5) satisfies R1, by Theorem 1, there exists some sufficiently smooth function $\mathbf{x}_c(w)$ for $w \in W$ with $\mathbf{x}_c(0) = 0$ satisfying (29). We need to show that satisfaction of R2 by the closed-loop system (5) implies that the function $\mathbf{x}_c(w)$ satisfies (30). In fact, with the notation used in the proof of Theorem 1, we can define a *normal* system as

$$\left.\begin{aligned} \dot{\bar{x}}_1 &= (\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}\bar{A}_{21})\bar{x}_1 + \bar{\phi}_3(\bar{x}_1, w) \\ \dot{w} &= s(w) \\ \bar{e} &= \bar{h}(\bar{x}_1, w) \end{aligned}\right\} \tag{33}$$

where

$$\bar{h}(\bar{x}_1, w) = h_c(M_{21}\bar{x}_1, M_{22}\alpha(\bar{x}_1, w), w) \tag{34}$$

and $M_2 = \left[M_{21} \mid M_{22}\right]$ with $M_{21} \in R^{n_c \times r}$. As shown in Theorem 1, (33) also satisfies R1, and there exists a sufficiently smooth function $\pi(w)$ that satisfies (21). We first claim that $\pi(w)$ also satisfies

$$0 = \bar{h}(\pi(w), w) \tag{35}$$

In fact, by (24) and (34), we have

$$\lim_{t \to \infty} h_c(x_c(t), w(t)) = \lim_{t \to \infty} \bar{h}(\bar{x}_1(t), w(t)) \tag{36}$$

Thus (33) satisfies the R2, or, what is the same, the trajectories of (33) starting from the sufficiently small initial states satisfy

$$\lim_{t \to \infty} \bar{h}(\bar{x}_1(t), w(t)) = 0 \tag{37}$$

since, by assumption, the closed-loop system (5) satisfies R2. We now recall from the output regulation theory for the *normal system* as can be found in Isidori and Byrnes (1990) that if, in addition to R1, (33) also satisfies R2, then $\pi(w)$ necessarily satisfies (35). Now noting that $\mathbf{x}_c(w)$ and $\pi(w)$ are related by (23) gives

$$h_c(\mathbf{x}_c(w), w) = \bar{h}(\pi(w), w) = 0 \tag{38}$$

That is, $\mathbf{x}_c(w)$ satisfies (30). □

Having established Lemma 1, it is possible to find out the solvability condition of the output regulation problem for both state feedback and output feedback as manifested in the following two theorems.

Theorem 2: *Under hypotheses* (H1) *and* (H2), *the output regulation problem is solvable by state feedback control if and only if there exist sufficiently smooth functions* $\mathbf{x}(w)$, *with* $\mathbf{x}(0) = 0$, *and* $\mathbf{u}(w)$, *with* $\mathbf{u}(0) = 0$, *both defined in a neighbourhood W of the origin of R^q satisfying the following:*

$$E \frac{\partial \mathbf{x}}{\partial w} s(w) = f(\mathbf{x}(w), \mathbf{u}(w), w) \tag{39}$$

$$h(\mathbf{x}(w), w) = 0 \tag{40}$$

Proof:

(Necessity) Assume the state feedback control $u = k(x, w)$ solves the state feedback output regulation problem. Then, by Lemma 1, there exists some sufficiently smooth function $\mathbf{x}_c(w)$ for $w \in W$ with $\mathbf{x}_c(0) = 0$ satisfying (29) and (30). Define $\mathbf{x}(w) = \mathbf{x}_c(w)$, and $\mathbf{u}(w) = k(\mathbf{x}(w), w)$. Then it is ready to verify that $\mathbf{x}(w)$ and $\mathbf{u}(w)$ satisfy (39) and (40).

For sufficiency, observe that, by Hypothesis (H2), there exists a matrix K such that $\sigma(E, A + BK) \in C^-$, and $\deg(\det(sE - (A + BK))) = \operatorname{rank}(E)$. Suppose conditions (39) and (40) are satisfied for some $\mathbf{x}(w)$ and $\mathbf{u}(w)$. Let

$$k(x, w) = \mathbf{u}(w) + K(x - \mathbf{x}(w))$$

This controller yields a closed-loop system with $x_c = x$, $E_c = E$, and $f_c(x_c, w) = f(x, k(x, w), w)$. Then, Requirement R1 is satisfied since the Jacobian matrix of $f_c(x_c, 0) = f(x, k(x, 0), 0)$ at the origin is equal to $A + BK$. Next, let $\mathbf{x}_c(w) = \mathbf{x}(w)$. Clearly, $k(\mathbf{x}_c(w), w) = \mathbf{u}(w)$. Thus (39) and (40) trivially reduce to (29) and (30). It follows from Lemma 1 that Requirement R2 is also fulfilled. □

Theorem 3: *Under Hypotheses* (H1), (H2) *and* (H3), *the output regulation problem via output feedback control is solvable if and only if there exist sufficiently smooth functions* $\mathbf{x}(w)$, *with* $\mathbf{x}(0) = 0$, *and* $\mathbf{u}(w)$, *with* $\mathbf{u}(0) = 0$, *both defined for* $w \in W$, *satisfying the conditions* (39) *and* (40).

Proof:

(Necessity) Assume the output feedback control $u = \beta(z, e)$, $E_z\dot{z} = g(z, e)$ solves the output regulation problem. Then, by Lemma 1, there exists some sufficiently smooth function $\mathbf{x}_c(w)$ for $w \in W$ with $\mathbf{x}_c(0) = 0$ satisfying (29) and (30). Perform the partition $\mathbf{x}_c(w) = \left[\mathbf{x}_1(w)^\tau \quad \mathbf{x}_2(w)^\tau\right]^\tau$ such that $\mathbf{x}_1(w) \in R^n$. Let $\mathbf{x}(w) = \mathbf{x}_1(w)$, and $\mathbf{u}(w) = \beta(\mathbf{x}_2(w), 0)$. Then it is ready to verify that $\mathbf{x}(w)$ and $\mathbf{u}(w)$ satisfy (39) and (40).

(Sufficiency) By assumption H2 and H3, there exist matrices H, G_1 and G_2 such that

$$(E, A + BH) \quad \text{and} \quad \left(\begin{bmatrix} E & 0 \\ 0 & I_q \end{bmatrix}, \begin{bmatrix} A - G_1C & P - G_1Q \\ -G_2C & S - G_2Q \end{bmatrix}\right) \quad \text{are strongly stable} \tag{41}$$

Suppose conditions (39) and (40) are satisfied for some $\mathbf{x}(w)$ and $\mathbf{u}(w)$. Let

$$\begin{bmatrix} E & 0 \\ 0 & I_q \end{bmatrix}\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} f(z_1, \mathbf{u}(z_2) + H(z_1 - \mathbf{x}(z_2)), z_2) - G_1\left[h(z_1, z_2) - e\right] \\ s(z_2) - G_2\left[h(z_1, z_2) - e\right] \end{bmatrix}$$

$$u = \beta(z, e) = \mathbf{u}(z_2) + H(z_1 - \mathbf{x}(z_2))$$

This controller yields a closed-loop system with $x_c = (x, z_1, z_2)$,

$$f_c(x_c, w) = \begin{bmatrix} f(x, \mathbf{u}(z_2) + H(z_1 - \mathbf{x}(z_2)), w) \\ f(z_1, \mathbf{u}(z_2) + H(z_1 - \mathbf{x}(z_2)), z_2) - G_1\left[h(z_1, z_2) - h(x, w)\right] \\ s(z_2) - G_2\left[h(z_1, z_2) - h(x, w)\right] \end{bmatrix}$$

and

$$E_c = \begin{bmatrix} E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & I_q \end{bmatrix}$$

The Jacobian matrix of $f_c(x_c, 0)$ at the origin is given as

$$A_c = \begin{bmatrix} A & BH & BK \\ G_1C & A + BH - G_1C & P + BK - G_1Q \\ G_2C & -G_2C & S - G_2Q \end{bmatrix}$$

where $K = \left[\partial\mathbf{u}/\partial z_2\right]_{z_2=0} - H\left[\partial\mathbf{x}/\partial z_2\right]_{z_2=0}$.

Some elementary transformation shows that

$$\det(sE_c - A_c) = \det(sE - (A + BH)) \times \det\left(\begin{bmatrix} E & 0 \\ 0 & I_q \end{bmatrix}s - \begin{bmatrix} A - G_1C & P - G_1Q \\ -G_2C & S - G_2Q \end{bmatrix}\right) \tag{42}$$

Thus (E_c, A_c) is also strongly stable. That is, requirement R1 is satisfied.

To verify R2, let $\mathbf{x}_c(w) = [\mathbf{x}(w)^T \ \mathbf{x}(w)^T \ w^T]$. Then it is clear that

$$h_c(\mathbf{x}_c(w), w) = h(\mathbf{x}(w), w) = 0 \tag{43}$$

$$\mathbf{u}(w) = \beta(\mathbf{x}(w), w, 0) \tag{44}$$

Using (43) and (44) and then (39) successively in the following gives

$$f_c(\mathbf{x}_c(w), w) = \begin{bmatrix} f(\mathbf{x}(w), \mathbf{u}(w), w) \\ f(\mathbf{x}(w), \mathbf{u}(w), w) \\ s(w) \end{bmatrix} = E_c \frac{\partial \mathbf{x}_c(w)}{\partial w} s(w)$$

That is, (29) and (30) are satisfied. □

Remark 5: It is seen that the solvability of the output regulation problem by both state feedback and output feedback control relies on the same set of equations given by (39) and (40). Clearly, this set of equations can be viewed as singular analogue of what is called regulator equations discovered by Isidori and Byrnes (1990). For convenience, we will refer to (39) and (40) as singular regulator equations in the sequel. We also note that the generalized Sylvester equation that governs the solvability of the output regulation problem for singular linear systems (Lin and Dai 1996) is a special case of (39) and (40).

5. Output regulation via normal output feedback controller

The output feedback controller constructed in Theorem 3 is also singular due to the singularity assumption on E. It is known that singular controllers are sensitive to the variations of initial conditions, and structured uncertainties. Moreover, it is less easy to realize singular controller physically. Thus it is desirable to synthesize *normal* controllers to solve our problem. It turns out that this is possible under an additional hypothesis:

(H4): (E, B) is normalizable, that is, there exists $L \in \mathbb{R}^{m \times n}$ such that $E + BL$ is nonsingular.

Theorem 4: *Under Hypotheses* (H1) *to* (H4), *the output regulation problem via a* normal *output feedback controller is solvable if and only if there exist sufficiently smooth functions* $\mathbf{x}(w)$, *with* $\mathbf{x}(0) = 0$, *and* $\mathbf{u}(w)$, *with* $\mathbf{u}(0) = 0$, *both defined in a neighbourhood* W *of the origin of* R^q *satisfying the conditions* (39) *and* (40).

Proof: The necessity can be shown in exactly the same way as in Theorem 3. For sufficiency, we first recall from Dai (1989) that Hypotheses (H2) and (H4) implies existence of the matrices H and L such that

$$E + BL \text{ is nonsingular and } (E + BL, A + BH) \text{ is strongly stable} \tag{45}$$

Also, Hypothesis (H3) implies existence of matrices G_1 and G_2 such that

$$\left(\begin{bmatrix} E & 0 \\ 0 & I_q \end{bmatrix}, \begin{bmatrix} A - G_1 C & P - G_1 Q \\ -G_2 C & S - G_2 Q \end{bmatrix} \right) \text{ is strongly stable} \tag{46}$$

Now suppose (39) and (40) hold for some $\mathbf{x}(w)$ and $\mathbf{u}(w)$. Let

$$\gamma(z, \dot{z}_1) = \mathbf{u}(z_2) + H\left[z_1 - \mathbf{x}(z_2)\right] - L\left[\dot{z}_1 - \frac{\partial \mathbf{x}}{\partial z_2} s(z_2)\right] \tag{47}$$

and

$$g_1(z, \dot{z}_1, e) = E\dot{z}_1 - f(z_1, \gamma(z, \dot{z}_1), z_2) + G_1\left[h(z_1, z_2) - e\right] \tag{48}$$

Then it can be verified that

$$\frac{\partial g_1}{\partial \dot{z}_1}(0,0,0) = E + BL$$

Since $E + BL$ is non-singular by (45), by the Implicit Function Theorem, there exists a sufficiently smooth locally defined function $\eta(z, e)$ satisfying $\eta(0,0) = 0$ such that

$$g_1(z, \eta(z, e), e) = 0 \tag{49}$$

Now define a *normal* output feedback control law as follows

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \eta(z, e) \\ s(z_2) - G_2\left[h(z_1, z_2) - e\right] \end{bmatrix}$$

$$u = \beta(z, e) = \gamma(z, \eta(z, e)) \tag{50}$$

which yields a closed-loop system with $x_c = \left[x^{\tau} \;\; z_1^{\tau} \;\; z_2^{\tau}\right]$,

$$E_c = \begin{bmatrix} E & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_q \end{bmatrix} \tag{51}$$

and

$$f_c(x_c, w) = \begin{bmatrix} f(x, \beta(z, h(x, w)), w) \\ \eta(z, h(x, w)) \\ s(z_2) - G_2\left[h(z_1, z_2) - h(x, w)\right] \end{bmatrix} \tag{52}$$

We will show that this control law solves the output regulation problem. To this end, let

$$N = \left.\frac{\partial \eta(z, h(x, 0))}{\partial (x, z)}\right|_{x=0, z=0} \tag{53}$$

Then using (49) shows that N satisfies

$$(E + BL)N = \left[G_1 C \quad A + BH - G_1 C \quad P + BK - G_1 Q\right] \tag{54}$$

where $K = \left[\partial \mathbf{u}/\partial z_2\right]_{z_2=0} - H\left[\partial \mathbf{x}/\partial z_2\right]_{z_2=0}$. Morever, we have

$$A_c \triangleq \left.\frac{\partial f_c(x_c, 0)}{\partial x_c}\right|_{x_c=0} = \begin{bmatrix} \left[A \quad BH \quad BK\right] - BLN \\ N \\ G_2 C \qquad -G_2 C \qquad S - G_2 Q \end{bmatrix} \tag{55}$$

We will first show that (E_c, A_c) is strongly stable. For this purpose, let

$$T = \begin{bmatrix} I_n & BL & 0 \\ 0 & E + BL & 0 \\ 0 & 0 & I_q \end{bmatrix}$$

Define

$$\bar{E}_c \stackrel{\text{def}}{=} TE_c = \begin{bmatrix} E & BL & 0 \\ 0 & E + BL & 0 \\ 0 & 0 & I_q \end{bmatrix}$$

$$\bar{A}_c \stackrel{\text{def}}{=} TA_c = \begin{bmatrix} A & BH & BK \\ G_1 C & A + BH - G_1 C & P + BK - G_1 Q \\ G_2 C & -G_2 C & S - G_2 Q \end{bmatrix}$$

Then, $(\bar{E}_c, \bar{A}_c)$ is strongly stable if and only if (E_c, A_c) is since T is non-singular. However, simple transformation gives

$$\det\left(s \begin{bmatrix} E & BL & 0 \\ 0 & E + BL & 0 \\ 0 & 0 & I_q \end{bmatrix} - \begin{bmatrix} A & BH & BK \\ G_1 C & A + BH - G_1 C & P + BK - G_1 Q \\ G_2 C & -G_2 C & S - G_2 Q \end{bmatrix} \right)$$

$$= \det(s(E + BL) - (A + BH)) \det\left(s \begin{bmatrix} E & 0 \\ 0 & I_l \end{bmatrix} - \begin{bmatrix} A - G_1 C & P - G_1 Q \\ -G_2 C & S - G_2 Q \end{bmatrix} \right).$$

Thus it follows from (45) and (46) that $(\bar{E}_c, \bar{A}_c)$, hence (E_c, A_c) is strongly stable, i.e. the closed-loop system satisfies R1.

Next we show the closed-loop system also satisfies R2, or, what is the same, equations (29) and (30) are satisfied for some $\mathbf{x}_c(w)$. To this end, again let $\mathbf{x}_c(w) = \left[\mathbf{x}(w)^\tau \mathbf{x}(w)^\tau \ w^\tau\right]$. Then equation (30) is trivially satisfied. Thus equation (29) can be expanded as

$$E \frac{\partial \mathbf{x}(w)}{\partial w} s(w) = f(\mathbf{x}(w), \beta(\mathbf{z}(w), 0), w) \tag{56}$$

$$\frac{\partial \mathbf{x}(w)}{\partial w} s(w) = \eta(\mathbf{z}(w), 0) \tag{57}$$

$$\frac{\partial w}{\partial w} s(w) = s(w) \tag{58}$$

where

$$\mathbf{z}(w) = \begin{bmatrix} \mathbf{x}(w) \\ w \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{z}_1(w) \\ \mathbf{z}_2(w) \end{bmatrix}, \quad \mathbf{z}_1(w) \in \mathbb{R}^n$$

Clearly, equation (58) is trivially satisfied. To verify (57), we note that $\eta(\cdot, \cdot)$ is uniquely defined by (49). Thus $\eta(\mathbf{z}(w), 0)$ is the unique solution of

$$g_1(\mathbf{z}(w), \eta(\mathbf{z}(w), 0), 0) = 0 \tag{59}$$

On the other hand, by the definition of $\eta(\cdot\,,\cdot)$, equation $\dot{z}_1 = \eta(z, e)$ is equivalent to $g_1(z, \dot{z}_1, e) = 0$, and also we have $\mathbf{z}_1(w) = \mathbf{x}(w)$. Thus, $(\partial \mathbf{x}(w)/\partial w)s(w)$ is the unique solution of

$$g_1\left(\mathbf{z}(w), \frac{\partial \mathbf{z}_1(w)}{\partial w}s(w), 0\right) = 0 \tag{60}$$

It follows from the uniqueness of the solution of both (59) and (60) that (57) holds. Finally we turn to (56) which is implied by (39) if $\mathbf{u}(w) = \beta(\mathbf{z}(w), 0)$ can be verified. But by the definition of $\beta(z, e)$ and using (57) leads to

$$\begin{aligned}\beta(\mathbf{z}(w), 0) &= \gamma(\mathbf{z}(w), \eta(\mathbf{z}(w), 0)) \\ &= \mathbf{u}(w) + H(\mathbf{x}(w) - \mathbf{x}(w)) - L\left[\frac{\partial \mathbf{x}(w)}{\partial w}s(w) - \frac{\partial \mathbf{x}(w)}{\partial w}s(w)\right] = \mathbf{u}(w)\end{aligned}$$

Thus, Requirement R2 is satisfied by Lemma 1. □

Remark 6: The *normal* control law (50) evolves from the control law

$$\begin{bmatrix} E & 0 \\ 0 & I_q \end{bmatrix}\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} f(z_1, u, z_2) - G_1[h(z_1, z_2) - e] \\ s(z_2) - G_2[h(z_1, z_2) - e] \end{bmatrix}$$
$$u = \gamma(z, \dot{z}_1) = \mathbf{u}(z_2) + H[z_1 - \mathbf{x}(z_2)] - L\left[\dot{z}_1 - \frac{\partial \mathbf{x}}{\partial z_2}s(z_2)\right] \tag{61}$$

which employs the technique of derivative feedback widely used in singular linear control problems, see, e.g. Lin and Dai (1996). The *normal* control law (50) follows from (61) by eliminating the reliance of u on $\dot{z}_1$ as exposed in the proof of the Theorem.

Remark 7: In case $f(x, u, w)$ is linear in u, i.e. $f(x, u, w) = f_1(x, w) + f_2(x, w)u$ for some sufficiently smooth $f_1(x, w)$ and $f_2(x, w)$. The implicit function $\eta(z, e)$ can be explicitly given by

$$\eta(z, e) = (E + f_2(z_1, z_2)L)^{-1}(f_1(z_1, \beta_1(z), z_2) - G_1[h(z_1, z_2) - e]) \tag{62}$$

6. Approximation solution of the singular regulator equations

Like *normal* systems, the key to the existence of either state feedback or output feedback controller is the solvability of the singular regulator equations (39) and (40). Due to the nonlinearity of the plant and exosystem, it is difficult to obtain the exact solution $\mathbf{x}(w)$ and $\mathbf{u}(w)$ for the singular or *normal* regulator equations. Thus an approximation method must be sought. Following the approach in Huang and Rugh (1992) that leads to a Taylor series solution of the *normal* regulator equations, we can also develop an approximation method for solving the singular regulator equations in the form of the Taylor series.

Our approach will involve power series representation for the unknown functions $\mathbf{x}(w)$ and $\mathbf{u}(w)$, and this entails the following notation. For any matrix K we will use the Kronecker product notation

$$K^{(0)} = 1, \quad K^{(1)} = K, \quad K^{(i)} = \underbrace{K \otimes K \cdots \otimes K}_{i_{factors}}, \quad i = 2, 3, \ldots \tag{63}$$

Then we can write the problem description in terms of the series expansions

$$\left.\begin{aligned} f(x,u,w) &= \sum_{l\geq 1}\sum_{\substack{i+j+k=l\\ i,j,k\geq 0}} F_{ijk}x^{(i)}\otimes u^{(j)}\otimes w^{(k)} \\ h(x,w) &= \sum_{l\geq 1}\sum_{\substack{i+k=l\\ i,k\geq 0}} H_{ik}x^{(i)}\otimes w^{(k)} \\ S(w) &= \sum_{i\geq 1} S_i w^{(i)} \end{aligned}\right\} \tag{64}$$

In order to obtain unique representations for the coefficients in series expansions of the unknown functions $\mathbf{x}(w)$ and $\mathbf{u}(w)$, the following notation will be used. For the $q\times 1$ vector w, let $w^{[l]}$ denote the vector

$$w^{[l]} = \begin{bmatrix} w_1^l & w_1^{l-1}w_2 & \cdots & w_1^{l-1}w_q & w_1^{l-2}w_2^2 & w_1^{l-2}w_2w_3 & \cdots & w_1^{l-2}w_2w_q & \cdots & w_q^l \end{bmatrix} \tag{65}$$

Then we seek series of the form

$$\mathbf{x}(w) = \sum_{k\geq 1} X_l w^{[k]}, \qquad \mathbf{u}(w) = \sum_{k\geq 1} U_l w^{[k]} \tag{66}$$

such that the singular regulator equations are satisfied formally. Note that the dimensions of $w^{[l]}$ and $w^{(l)}$ are, respectively,

$$\begin{bmatrix} q+l-1 \\ l \end{bmatrix}\times 1, \qquad q^l\times 1 \tag{67}$$

and that there exist matrices M_l and N_l of appropriate dimensions such that

$$w^{[l]} = M_l w^{(l)}, \qquad w^{(l)} = N_l w^{[l]} \tag{68}$$

Substituting (64) and (66) into (39) and (40) and identifying the coefficients of $w^{[l]}$, $l = 1,2,\ldots,$ yields the following result.

Lemma 2: *The power series* (66) *formally satisfy the singular regulator equations* (39) *and* (40) *if and only if the following generalized Sylvester equation is satisfied for* $l = 1,2,\ldots$:

$$\left.\begin{aligned} EX_lM_l\left[\sum_{i=1}^{l} I_q^{(i-1)}\otimes S\otimes I_q^{(l-i)}\right]N_l &= AX_l + BU_l + P_l \\ 0 &= CX_l + Q_l \end{aligned}\right\} \tag{69}$$

where

$$A = F_{100},\ B = F_{010},\ P_1 = P = F_{001}$$
$$C = H_{10},\ Q_1 = Q = H_{01},\ S = S_1$$

and, for $l = 2,3,\ldots,$

$$
\begin{aligned}
P_l = P &+ \sum_{n=2}^{l} \sum_{\substack{i+j+k=l \\ i,j,k \geq 0}} F_{ijk} G_{l-n}^{ij} N_l \\
&- \sum_{k=1}^{l-1} X_k M_k \left[\sum_{\substack{j_1+j_2+\cdots+j_k=l \\ j_1,j_2,\cdots,j_k \geq 1}} S_{j_1} \otimes S_{j_2} \otimes \cdots \otimes S_{j_k} \right] N_l
\end{aligned} \tag{70}
$$

$$
Q_l = Q + \left[\sum_{n=2}^{l} \sum_{\substack{i+k=l \\ i,k \geq 0}} H_{ik} G_{l-n}^{ij} \right] N_l \tag{71}
$$

$$
G_m^{ij} = \begin{cases} 0, & i = j = 0, m > 0, \\ 1, & i = j = 0, m > 0, \\ \delta_{i,j+m}, & j = 0, i = 1,2,\ldots, \\ \lambda_{i,j+m}, & j = 0, i = 1,2,\ldots, \\ \sum_{k=0}^{m} \delta_{i,i+k} \otimes \lambda_{j,j+m-k}, & i,j = 1,2,\ldots, \end{cases}
$$

$$
\delta_{i,j} = \sum_{\substack{j_1+j_2+\cdots+j_i=l \\ j_1,j_2,\cdots,j_i \geq 1}} X_{j_1} M_{j_1} \otimes X_{j_2} M_{j_2} \otimes \cdots \otimes X_{j_i} M_{j_i} \tag{72}
$$

$$
\lambda_{i,j} = \sum_{\substack{j_1+j_2+\cdots+j_i=l \\ j_1,j_2,\cdots,j_i \geq 1}} U_{j_1} M_{j_1} \otimes U_{j_2} M_{j_2} \otimes \cdots \otimes U_{j_i} M_{j_i} \tag{73}
$$

Proof: The proof is quite similar to that given in Lemma 5.1 of Huang and Rugh (1992), and is therefore omitted. □

Note that X_l and U_l depend only on $X_1, \ldots, X_{l-1}$ and $U_1, \ldots, U_{l-1}$, so that (69) provides an iterative sequence of the generalized Sylvester equations. The following result establishes the solvability condition for these equations.

Theorem 5: *There exists a solution (unique if $p = m$) of (69) for any P_l and Q_l, $l = 1,2,\ldots,$ if and only if*

$$
\operatorname{rank} \begin{bmatrix} A - \lambda E & B \\ C & 0 \end{bmatrix} = n + p \tag{74}
$$

for all λ given by

$$
\lambda = \lambda_{i_1} + \cdots + \lambda_{i_l} \tag{75}
$$

where $i_1, \cdots, i_l \in \{1,2,\cdots,q\}$ and $\lambda_1, \cdots, \lambda_q$ are eigenvalues of S.

Proof: Being a generalized Sylvester equation, (69) is solvable for any (P_l, Q_l) if and only if the rank condition (74) holds for all λ in the spectrum of the matrix

$$M_l \left[\sum_{i=1}^{l} I_q^{(i-1)} \otimes S \otimes I_q^{(l-i)} \right] N_l \tag{76}$$

However, it is shown that the spectrum of the matrix (76) is precisely that described in the statement of Theorem 5 Huang and Rugh (1992). □

7. Concluding remarks

This paper has extended some existing results of the output regulation problem for either singular linear systems or *normal* nonlinear systems to singular nonlinear systems. A few remarks are in order. First, it is possible to relax Hypotheses H2 and H3 from *strongly stabilizable* to plain *stabilizable* at the cost of resulting in a closed-loop system whose linearization is *stable* rather than *strongly stable*. The response of such systems is known, however, to exhibit impulsive behaviour. Second, our result on designing *normal* output feedback controller relies on the normalizability assumption of (E, B). It may be interesting to look into the possibility of removing the normalizability assumption.

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